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ABSTRACT

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A method was proposed in reference 1 for solving the variational problem of maximum useful load delivery from a motor with limited power in a given operating time. The essence of this method is as follows: a new phase coordinate  $t_m$  is introduced which is the running time of the motor, as well as a new control  $\delta(t)$ --a relay function which takes on the value 1 when the motor is "on", and 0 when the motor is "off"--and a differential relationship is found between  $t_m(t)$  and  $\delta(t)$ :  $t_m = \delta$ . The control function of the motor--the discharge (or thrust)--is multiplied in all equations by the function  $\delta$  so that the new discharge coincides with the old when  $\delta = 1$ , and vanishes when  $\delta = 0$ . The equation for  $\delta(t)$  is determined from conditions for the extremum of the central functional and boundary conditions for  $t_m(0) = 0$ ,  $t_m(T) = T_m$ . For plane-parallel and zero-force fields, the variational problem is completely solved, and results are given in reference 1 for setting up a given modulus of speed and flight between two quiescent points.

A similar formulation of the variational problem for a

\*Numbers given in margin indicate pagination in original foreign text.

over

central field requires a numerical solution. In this paper the authors solve the variational problem for flight between circular orbits in a central field. A motor with limited power is assumed which is ideally controllable. The operating time of the motor is given and is less than the time of motion. In section 1, the problem is formulated and reduced to a boundary problem for ordinary differential equations. The method for solving this boundary problem is set forth in section 2, and the results of calculations are given in section 3.

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1. Equations of plane motion in a central field in terms of acceleration are given as follows:

$$\begin{aligned} x'' &= a \cos \theta - fMx / (x^2 + y^2)^{3/2} \\ y'' &= a \sin \theta - fMy / (x^2 + y^2)^{3/2}. \end{aligned} \quad (1.1)$$

Here  $x$  and  $y$  are rectangular coordinates with the origin at the center of the gravitational field;  $f$  is the gravitational constant;  $M$  is the mass of the central body;  $a$  is reactive acceleration;  $\theta$  is the angle between the thrust vector and axis  $Ox$ ; and a dot indicates differentiation with respect to time.

The initial and final values of coordinates  $x(0)$ ,  $y(0)$ ,  $x(T)$ ,  $y(T)$  and velocities  $\{\dot{x}(0)$ ,  $\dot{y}(0)$ ,  $\dot{x}(T)$ ,  $\dot{y}(T)\}$  for motion between two circular orbits may be expressed in terms of the radii of the orbits  $r_0$ ,  $r_1$  and the angular displacement  $\phi_1$

$$\begin{aligned} x(0) &= r_0, & y(0) &= 0, & \dot{x}(0) &= 0, & \dot{y}(0) &= \sqrt{fM/r_0} \\ x(T) &= r_1 \cos \phi_1, & y(T) &= r_1 \sin \phi_1 \\ \dot{x}(T) &= -\sqrt{fM/r_1} \sin \phi_1, & \dot{y}(T) &= \sqrt{fM/r_1} \cos \phi_1. \end{aligned} \quad (1.2)$$

The energy value of a maneuver for an ideally controllable motor of limited power is given by the functional

$$J = \int_0^T a^2 dt \quad (1.3)$$

The useful load is a maximum when  $J$  is a minimum.

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To solve the variational problem for a minimum of functional  $J$  when the time  $T_m$  for effecting control is given, we introduce the relay function  $\delta(t)$ , and effective running time  $t_m(t)$ . The complete system of equations for the problem is (ref. 1):

$$\begin{aligned} J' &= a^2 \delta, \quad \dot{x} = u, \quad \dot{y} = v, \quad \dot{t}_m = \delta \\ u &= a \delta \cos \vartheta - x / (x^2 + y^2)^{3/2}, \quad v = a \delta \sin \vartheta - y / (x^2 + y^2)^{3/2}, \\ J(0) &= 0, \quad x(0) = 1, \quad y(0) = 0, \quad u(0) = 0, \quad v(0) = 1, \quad t_m(0) = 0 \\ J(T) &\rightarrow \min, \quad x(T) = r_1 \cos \varphi_1, \quad y(T) = r_1 \sin \varphi_1 \\ u(T) &= -(\sin \varphi_1) / r_1^{1/2}, \quad v(T) = \cos \varphi_1 / r_1^{1/2}, \quad t_m(T) = T_m \end{aligned} \quad (1.4)$$

In system (1.4), the times, coordinates, accelerations and functional appearing in expressions (1.1), (1.2) and (1.3) are de-dimensionalized: The linear dimensions are adjusted to the radius of the initial orbit  $r_0$ , times  $t$  and  $t_m$  are reduced to the time of revolution in the initial orbit divided by  $2\pi$ , and the old notation is maintained for dimensionless values. Differential equations of the second order (1.1) are given as four equations of the first order.

To find optimum controls  $a(t)$ ,  $\theta(t)$ ,  $\delta(t)$ , we set up the Hamiltonian and write out the equations for the momenta

$$\begin{aligned} H &= -a^2 \delta + p_x u + p_y v + p_u [a \delta \cos \vartheta - x / (x^2 + y^2)^{3/2}] + \\ &\quad + p_v [a \delta \sin \vartheta - y / (x^2 + y^2)^{3/2}] + p_m \delta \\ p_x' &= \frac{p_u}{(x^2 + y^2)^{3/2}} - \frac{3x(p_u x + p_v y)}{(x^2 + y^2)^{5/2}}, \quad p_y' = \frac{p_v}{(x^2 + y^2)^{3/2}} - \frac{3y(p_u x + p_v y)}{(x^2 + y^2)^{5/2}} \\ p_u' &= -p_x, \quad p_v' = -p_y, \quad p_m' = 0 \end{aligned} \quad (1.5)$$

The maximum for  $H$  with respect to  $a$ ,  $\theta$  and  $\delta$  is determined by fulfillment of the following conditions:

$$\begin{aligned}\cos \vartheta &= \frac{p_u}{\sqrt{p_u^2 + p_v^2}}, \quad \sin \vartheta = \frac{p_v}{\sqrt{p_u^2 + p_v^2}} \\ a &= 1/2 \sqrt{p_u^2 + p_v^2}, \quad \delta = 1 \text{ when } \Delta > 0, \quad \delta = 0 \text{ when } \Delta < 0 \\ (\Delta &= 1/4 (p_u^2 + p_v^2) + p_m)\end{aligned}\quad (1.6)$$

Thus, the variational problem has been reduced to solving the boundary problem for the differential equations

$$\begin{aligned}\dot{x} &= u, \quad \dot{y} = v, \quad \dot{u} = p_u \delta - x / (x^2 + y^2)^{1/2} \\ \dot{v} &= p_v \delta - y / (x^2 + y^2)^{1/2}, \quad \dot{t}_m = \delta \\ p_x &= \frac{p_u}{(x^2 + y^2)^{1/2}} - \frac{\partial (p_u x + p_v y)}{(x^2 + y^2)^{3/2}}, \quad p_y = \frac{p_v}{(x^2 + y^2)^{1/2}} - \frac{3y(p_u x + p_v y)}{(x^2 + y^2)^{3/2}} \\ p_u &= -p_x, \quad p_v = -p_y, \quad p_m = 0 \\ \delta &= 1 \text{ when } \Delta > 0, \quad \delta = 0 \text{ when } \Delta < 0 \quad (\Delta = 1/4 (p_u^2 + p_v^2) + p_m, \quad p_m < 0) \\ x(0) &= 1, \quad y(0) = 0, \quad u(0) = 0, \quad v(0) = 1, \quad t_m(0) = 0 \\ x(T) &= r_1 \cos \varphi_1, \quad y(T) = r_1 \sin \varphi_1, \quad t_m(T) = T_m \\ u(T) &= -(\sin \varphi_1) / r_1^{1/2}, \quad v(T) = \cos \varphi_1 / r_1^{1/2}.\end{aligned}\quad (1.7)$$

After solution of the boundary problem, the functions

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$x(t), y(t), \dots, p_u(t), p_v(t), p_m$  are determined, and the control functional

$$J = \int_0^T a^2 \delta dt = 1/4 \int_0^T (p_u^2 + p_v^2) \delta dt \quad (1.8)$$

may be calculated.

The variational problem under consideration contains four parameters:  $T, \phi_1, r_1$  and  $T_m$ . If the conditions of the problem are such that the parameter  $T_m$  must run through a number of values ( $T_m \leq T$ ), then it is convenient to carry out the parametric calculations with respect to  $p_m$ , and not with respect to  $T_m$ . In this case, there is no necessity in (1.7) to satisfy the condition  $t_m(T) = T_m$ ; the equation for  $t_m$  may be integrated after solution of the boundary problem simultaneously with the calculation of functional (1.8)

$$T_m = \int_0^T \delta(t) dt. \quad (1.9)$$

2. When solving boundary problem (1.7) (with parameter  $p_m$  and not  $T_m$ ), the values of  $x(T), y(T), u(T), v(T)$  must be chosen in such a way that the functions  $\sigma_x, \sigma_y, \sigma_u, \sigma_v$  are equal to the given values at the end of the interval on the right. Let us form difference-discrepancies between the given values of the coordinates and velocities and the values derived for a fixed set of initial values

$$p_{x0}, p_{y0}, p_{u0}, p_{v0}$$

$$\begin{aligned} \sigma_x(p_{x0}, p_{y0}, p_{u0}, p_{v0}) &= r_1 \cos \varphi_1 - x_1(p_{x0}, p_{y0}, p_{u0}, p_{v0}) \\ \sigma_y(p_{x0}, p_{y0}, p_{u0}, p_{v0}) &= r_1 \sin \varphi_1 - y_1(p_{x0}, p_{y0}, p_{u0}, p_{v0}) \\ \sigma_u(p_{x0}, p_{y0}, p_{u0}, p_{v0}) &= -\sin \varphi_1 / r_1^{1/2} - u_1(p_{x0}, p_{y0}, p_{u0}, p_{v0}) \\ \sigma_v(p_{x0}, p_{y0}, p_{u0}, p_{v0}) &= \cos \varphi_1 / r_1^{1/2} - v_1(p_{x0}, p_{y0}, p_{u0}, p_{v0}) \end{aligned} \quad (2.1)$$

For each quadrupole  $(p_{x0}, p_{y0}, p_{u0}, p_{v0})$ , we may compute the values of the functions  $\sigma_x, \sigma_y, \sigma_u, \sigma_v$  and consequently the discrepancies  $\sigma_x, \sigma_y, \sigma_u, \sigma_v$  if we solve the Cauchy boundary problem for system (1.7) with the deficient initial values

By calculating the functions  $\sigma_x, \sigma_y, \sigma_u, \sigma_v$  at any point in the space and applying the rules for numerical differentiation, we can find the first derivatives  $\partial \sigma_x / \partial p_{x0}, \partial \sigma_x / \partial p_{y0}, \dots, \partial \sigma_y / \partial p_{x0}, \dots$  second derivatives, etc.

Selection of the deficient initial values of  $p_{x0}, p_{y0}, p_{u0}, p_{v0}$  which satisfy the final conditions for coordinates and velocities reduces to solving a system of algebraic equations given in implicit form

$$\begin{aligned} \sigma_x(p_{x0}, p_{y0}, p_{u0}, p_{v0}) &= 0, & \sigma_y(p_{x0}, p_{y0}, p_{u0}, p_{v0}) &= 0 \\ \sigma_u(p_{x0}, p_{y0}, p_{u0}, p_{v0}) &= 0, & \sigma_v(p_{x0}, p_{y0}, p_{u0}, p_{v0}) &= 0 \end{aligned} \quad (2.2)$$

Newton's method was used for finding the roots of equation (2.2). The algorithm of this method is quite well known and is frequently used in equations of this type; therefore we shall point out only those special points in

the method associated with the large number of parameters in the problem.

We shall assume that the boundary problem is solved for the parameters  $T', \phi_1', r_1', T_m'$ , i. e., the initial deficient values  $p_{x0}', p_{y0}', p_{u0}', p_{v0}'$  have been determined, and the problem must be solved for  $T', \phi_1', r_1', T_m''$ . If the parameter  $T_m''$  differs only slightly from  $T_m'$ , then it is appropriate to take the previous solution  $p_{x0}, p_{y0}, p_{u0}, p_{v0}$  as the zero approximation of Newton's method for  $p_{x0}', p_{y0}', p_{u0}', p_{v0}'$ . The closer the values of  $T_m'$  and  $T_m''$ , the higher the probability that Newton's iteration process will be convergent with this zero approximation.

The problem may be checked with respect to parameter  $T_m$  down to /59 small values  $T_m \leq T$  by making any small change in the parameter  $T_m$  so that the previous solution  $p_{x0}, p_{y0}, p_{u0}, p_{v0}$  may be used to assure a convergent process. A similar argument holds for the other parameters  $T, \phi_1$  and  $r_1$ .

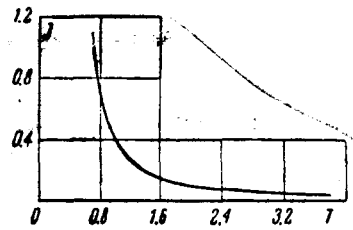


Figure 1

If  $T \ll 1$ ,  $\phi_1 \ll 1$ ,  $T_m \neq T$ , the central field is satisfactorily approximated by a zero-force field, and the variational problem has a simple analytical solution (ref. 1). By finding  $p_{x0}, p_{y0}, p_{u0}, p_{v0}$  from this solution, we may use them as a zero approximation for the problem in a central field. Then by successively changing the parameters  $T, \phi_1, r_1, T_m$  to higher and lower values as necessary and using the procedure described above, the problem may be checked in the required range for  $T, \phi_1, r_1, T_m$ .

Selection of the initial unknowns  $p_{x0}, p_{y0}, p_{u0}, p_{v0}$  is assumed to be com-

pleted when the condition

$$\sqrt{(\sigma_x^{(n)})^2 + (\sigma_y^{(n)})^2 + (\sigma_u^{(n)})^2 + (\sigma_v^{(n)})^2} \leq \varepsilon \quad (2.3)$$

is fulfilled at the  $n$ -th step of the process. Here  $\varepsilon$  is the permissible error in calculation.

3. Calculations were made for flight from a circular orbit with radius  $r_0 = 1$  to a circular orbit with  $r_1 = 1.52$ ; in most cases, the mean angular velocity of the flight was taken as equal to the angular velocity of motion in the initial orbit:  $(\phi_1/T) = 1$ .

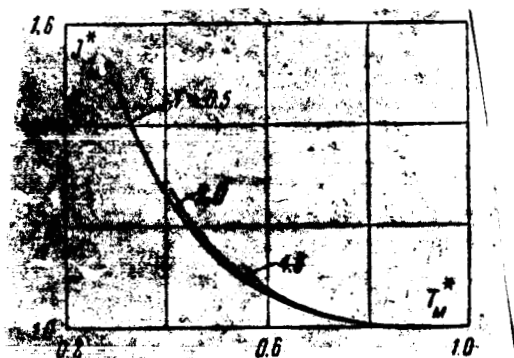


Figure 2



Figure 3

The relationship between the functional  $J$  and time of motion  $T$  is given in figure 1 for the case where there are no passive regions on the trajectory ( $T_m = \phi_1 = T$ ). Figure 2 shows the family of curves  $J^*(T_m^*)$  for various values of  $T(\phi_1 = T)$ , where

$$J^* = J(T_m, T) / J_{T_m=T}, \quad T_m^* = T_m / T$$

The calculations showed that there is one passive region on the trajectory in the range  $0.5 \leq T \leq 4.5$ ; in this interval of variations in times of motion, and for  $0.8 < T_m^* \leq 1.0$ , the relative change in the functional--the quantity  $J^*$ --is weakly dependent on  $T$ . When  $T_m^* = 0.5$ , the relative change in the functional for various  $T$  is 12-13%, which agrees satisfactorily with

the results for a zero-force field. Laws for the change in the modulus of reactive acceleration are given in figure 3 for  $T = 4.5$  and various values of  $T_m$ .

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